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Performance and optimum dimensions of flat fins for tube-and-fin heat exchangers: A generalized analysis

B. Kundu [a,*], P.K. Das [b]

[a] Department of Mechanical Engineering, Jadavpur University, Kolkata 700 032, India
[b] Department of Mechanical Engineering, Indian Institute of Technology Kharagpur, Kharagpur 721 302, India

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ABSTRACT

The performance of flat fins for tube-fin heat exchangers has been analyzed for both inline and staggered arrangement of tubes. In earlier published studies, regular square and equilateral triangular array tube layouts were considered. No such restriction is put in the present study. The analysis has been done by a semi-analytical technique where the boundary condition at the fin edge is discretely satisfied at a large number of points by a method of collocation. It has also been demonstrated that the approximate results obtained by the sector method closely agree with the prediction of semi-analytical technique. Finally, a generalized scheme of optimization based on Lagrange multiplier technique has been suggested which shows that irrespective of the volume and thickness of the fins, square and equilateral triangular array of tubes are the optimum layout for inline and staggered arrangements, respectively. This result was known so far only intuitively. The optimum dimensions for flat fins for other layout of tubes have also been obtained specifying the ratio of longitudinal to transverse tube pitch.

1. Introduction

The rate of heat exchange between a gas and liquid stream is limited due to the low heat transfer coefficient of the gas side. External fins attached on the liquid carrying tubes circumvent this problem to a large extent. Individual fins of different geometry or integral tube-fin heat exchangers are used in practice. Due to their compactness tube-fin heat exchangers find wide applications in HVAC systems, power plants and process industries. In such cross-flow heat exchangers, the gas stream flows over flat fins. The liquid stream flows through the tubes, which pierce the flat fins in a regular array, and then generally tubes are mechanically expanded on to fins for a good tube-to-fin thermal contact. The tubes may have inline or staggered arrangement as shown in Fig. 1. The geometry of the inline arrangement can be specified by the longitudinal pitch (P_l) and transverse pitch (P_t). For staggered arrangement, one can define the transverse pitch (P_t) and the diagonal pitch (P_d). In both of these arrangements, the flat fins can be divided into a number of repetitive modules, which circumscribe a particular tube and have adiabatic boundaries. For inline arrangement this is a rectangle, while in staggered arrangement this is an irregular hexagon. The rate of heat exchange from these polygonal fins needs to be analyzed for predicting the performance of flat fin and round tube heat exchangers.

It may be noted that in each of the above arrangements if the pitch lengths are made equal, i.e. $P_l = P_t$ for inline tubes and $P_t = P_d$ for staggered tubes, a square array is obtained in the former case and an equilateral triangular array is formed in the later case. Each tube then dissipates heat through a square fin or a regular hexagonal fin respectively.

The difficulty of analyzing circumferential polygonal fins arises from the fact that the temperature field in the fins is two-dimensional and the fin geometry conforms neither to the rectangular Cartesian coordinate system nor to a cylindrical polar coordinate. A closed form expression for the fin efficiency is not available. Sparrow and Lin (1964) analyzed the heat transfer from circumferential fins of square and hexagonal geometry by an entirely different approach, which they termed as the *semi-analytical technique*. Using separation of variables technique, they obtained the temperature distribution in a series form. They satisfied the adiabatic condition at the fin tip only at discrete points and determined the coefficients of the series by a collocation method. Zabronski (1955) gave an analytical solution for the temperature distribution and efficiency of square fins around circular tubes. In this analysis, the adiabatic boundary condition at the fin edge has been satisfied exactly while the isothermal condition at the fin base was satisfied only approximately. These approximate methods of analysis have only a limited application in practical design. Parallel efforts have been made to find out approximate analytical techniques such that expression for fin performance is obtained by using the efficiency of circular fins.

* Corresponding author. Tel.: +91 9433031203; fax: +91 33 24146890.
E-mail address: bkundu123@rediffmail.com (B. Kundu).

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Nomenclature

a	area of the symmetric sector (m^2)
A	dimensionless area, $a/2r_i^2$
a_n	area of nth sector (m^2)
A_n	area of nth sector, dimensionless
B_1,B_2	variables expressed in Eqs. (9) and (10), dimensionless
Bi	Biot number, hr_i/k, dimensionless
C_j	constant (j = 0, 1, 2,...), see Eq. (6), dimensionless
D_1, D_2	variables defined in Eqs. (18) and (19), dimensionless
f_1, f_2, f_3	functions, see Eqs. (27) and (29), dimensionless
F	factor (1 for inline array, 2 for staggered array)
g	notation expressed in Eq. (28), dimensionless
h	convective heat transfer coefficient (W m^{-2} K^{-1})
$I_n(Z)$	modified Bessel function of first kind and order n having argument Z
J	Jacobian matrix, dimensionless
k	thermal conductivity of the fin material (W m^{-1} K^{-1})
$K_n(Z)$	modified Bessel function of second kind and order n having argument Z
m	total number of sectors
n	integer number
p	total number of points taken on the fin tip
P_d	diagonal pitch (m)
P_l	longitudinal pitch (m)
P_t	transverse pitch (m)
q	actual heat transfer rate (W)
Q	actual heat transfer rate, defined in Eq. (11), dimensionless
q_e	heat transfer rate, see Eq. (14) (W)
Q_e	heat transfer rate, defined in Eq. (14), dimensionless
q_i	ideal heat transfer rate (W)
Q_i	ideal heat transfer rate, see Eq. (12), dimensionless
r	radial distance (m)
r_i	outer radius of tubes (m)
r_t	tip distance from the tube center (m)
R	r/r_i, dimensionless
R_a	s_l/s_t, dimensionless
R_t	r_t/r_i, dimensionless
R_{tn}	dimensionless radial distance, see Eq. (21)
s_l	longitudinal distance of the symmetric sector, see Fig. 1 (m)
s_t	transverse distance of the symmetric sector, see Fig. 1 (m)
S_l	s_l/r_i, dimensionless
S_t	s_t/r_i, dimensionless
t	semi-thickness of the fin (m)
T	t/r_i, dimensionless
T_a	ambient fluid temperature (K)
T_b	temperature at the outer tube (K)
T_f	local fin temperature(K)
U	fin volume, see Eq. (25), dimensionless
v	fin volume (m^3)
Z_0	fin parameter, $\sqrt{\mathrm{Bi}/T}$, dimensionless

Greek letters

α_0	$\tan^{-1}(S_t/S_l)$, dimensionless
φ	angular position in the plate, deg
φ_n	defined in Eq. (23), deg
ε	fin effectiveness, see Eqs. (15) and (24), dimensionless
η	fin efficiency, see Eqs. (13) and (16), dimensionless
η_n	fin efficiency of nth sector, see Eq. (17), dimensionless
θ	temperature, $(T_f - T_a)/(T_b - T_a)$, dimensionless
Δ_1, Δ_2, Δ_3	notations used in Eq. (33), dimensionless
Γ_1, Γ_2, Γ_3	notations used in Eq. (34), dimensionless
$\mathbf{\Omega}_1$, $\mathbf{\Omega}_2$, $\mathbf{\Omega}_3$	matrixes expressed in Eqs. (35)–(37), dimensionless

Equivalent annulus technique is the simplest of the approximate techniques. In this method, the efficiency of the polygonal fin is considered equal to that of a circular fin having identical inner radius, fin thickness and fin surface area (Zabronski, 1955). Carrier and Anderson (1944) pointed out the limitations of the equivalent annulus method. Sparrow and Lin (1964) have also demonstrated the inaccuracy of results predicted by this method for a certain range of fin geometry. Kuan et al. (1984) conducted a numerical study and determined the efficiency of a variety of polygonal fins circumscribing tubes of different geometry. They found that the equivalent annulus method is inaccurate when the tube or the fin geometry becomes elongated.

An improvement over the equivalent annulus method is the sector method (Shah, 1985). In this method, the polygonal fin is divided into a number of radial sectors. The approximate efficiency of each sector is determined assuming it to be a part of an annular fin. Finally, the weighted average of the sector fin efficiency yields the fin efficiency of the flat fin.

Kundu and Das (2000) compared the accuracy of prediction of these two approximate techniques with the semi-analytical technique of Sparrow and Lin (1964) for square, hexagonal and eccentric annular fins. They reported that the accuracy of the equivalent annulus method decreases with the increase of two-dimensionality of the fins. In fact, the equivalent annulus method cannot be applied for eccentric annular fins. On the other hand, the sector method shows a close agreement with the semi-analytical technique over a wide range of thermo-geometric parameters of the fins. Kundu and Das (2000a) have further provided a modified formula of the sector method to analyze polygonal fins with tip heat loss.

In a separate work, Kundu and Das (2000b) suggested another improvisation of the sector method. They have defined the outer radius of the sector based on its area. This modification unifies the equivalent annulus method and the sector method. Moreover, a better accuracy is obtained with a fewer number of sectors.

McQuiston et al. (2001) suggested the use of empirical formulation originally proposed by Schmidt (1945-46) for determining the equivalent outer radius of polygonal fins. For inline and staggered arrangement of tubes, separate relationships as functions of the relevant geometrical parameters have been given.

While the addition of surface area due to fins in a heat exchanger is essential to compensate for low heat transfer coefficient of gas/oil flowing over fins, addition of the fins increases weight, volume and the initial cost of heat exchangers. In tube-fin heat exchangers, the tube spacing and fin thickness can be selected optimally so that the maximum heat can be transferred for a given fin volume. Starting from the semi-analytical technique of Sparrow and Lin (1964), Kundu and Das (1997) determined the optimum dimensions of square and regular hexagonal fins under the constraint of specified fin volume. Kundu and Das (1999) further extended this study to determine the optimum dimensions of eccentric annular disc fins. Kundu and Das (2000a) also demonstrated that the optimum dimensions of square and hexagonal fins can be predicted very accurately by the sector method. Perrotin and Clodic (2003) determined the fin efficiency in enhanced fin-and-tube heat exchangers in dry conditions. The performance characteristics correlation for round tube and plate finned heat exchangers was determined experimentally by Abu Madi et al. (1998).

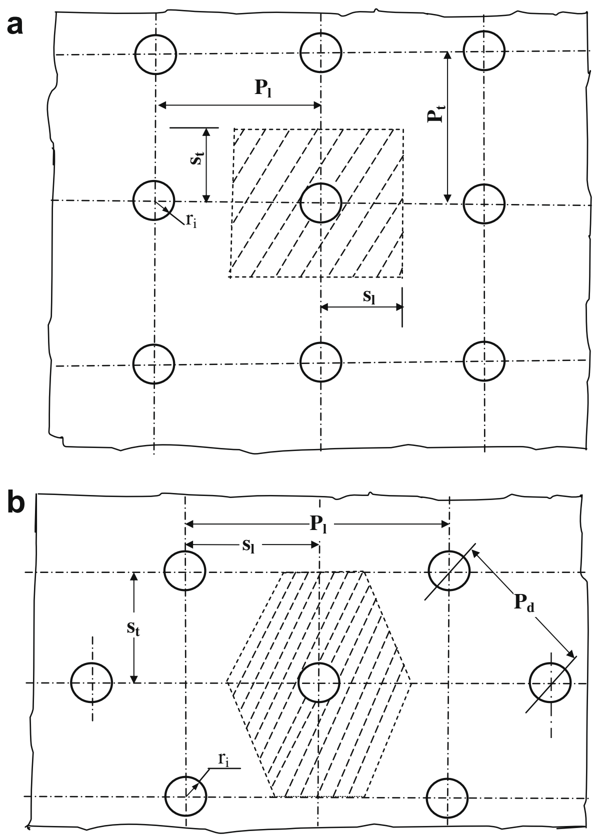

Fig. 1. Schematic representation of fin tube heat exchangers: (a) inline arrangement; (b) staggered arrangement.

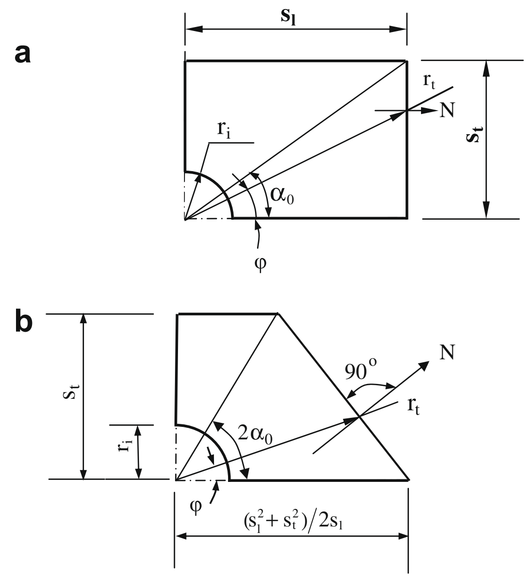

Fig. 2. Symmetric section of an irregular polygonal fin: (a) inline arrangement; and (b) staggered arrangement.

The above discussion shows that most of the previous work considered regular polygonal fins which pertain to only square array and equilateral triangular array of tubes. Nevertheless, tube layouts different from these are often used for various practical reasons. Though Shah (1985) suggested the use of sector method and Schmidt (1945-46) proposed empirical formulas for irregular polygonal fins, no analysis for such geometry has so far been forwarded. In the present paper, generalized layouts (as shown in Fig. 1) for inline and staggered arrangement of tubes have been considered. The symmetrical module surrounding each of the tubes are rectangle and irregular hexagon respectively in these two cases. Taking the queue from the analysis of Sparrow and Lin (1964), the temperature distribution, fin efficiency and fin effectiveness of these two polygonal fins have been determined using a collocation method. It has also been demonstrated that the fin performance predicted by the sector method agrees closely with the analytical results. Finally, a generalized formulation has been made for determining the optimum dimension of fins and a method has been suggested for constructing the design curves for optimum fins.

2. Analysis

In the polygonal fins shown in Fig. 1a and b, two lines of symmetry may be identified. These zero temperature gradient lines intersect each other orthogonally at the tube center and deviates each fin into four thermally symmetric sections. It is suffice to analyze any of the sections for determining the performance of the whole fin. In Fig. 2, the symmetric sections of the rectangular and irregular hexagonal fins have been depicted.

The following idealizations are made for the analysis:

1. Heat exchange between the fin and the surrounding fluid takes place in steady state.
2. The heat transfer from the fin surface to environment is solely by convection.
3. The conductivity of fin material is uniform and constant.
4. The convective heat transfer coefficient and the temperature of the fluid medium are constant and uniform.
5. Though there is temperature variation both along radial and azimuthal directions, the temperature gradient normal to the fin surface is neglected, as its thickness is small compare to other dimensions.
6. The thermal contact resistance between the fin and the tube is negligible.
7. Longitudinal heat conduction in the fin along the fluid flow direction is negligible.

With the above idealizations, the governing equation for the flat fins can be written in the cylindrical polar coordinate as follows:

$$\frac{\partial}{\partial R}\left(R\frac{\partial\theta}{\partial R}\right)+\frac{1}{R}\frac{\partial^2\theta}{\partial\phi^2}=Z_0^2R\theta \qquad (1)$$

where

$$\begin{bmatrix}\theta\\ Z_0\\ R\end{bmatrix}=\begin{bmatrix}(T_f-T_a)/(T_b-T_a)\\ \sqrt{Bi/T}\\ r/r_i\end{bmatrix} \qquad (2)$$

The above differential equation is subjected to the following boundary conditions:

$$\theta=1 \quad \text{at } R=1 \text{ and for } 0\le\phi\le\pi/2 \qquad (3a)$$

$$\frac{\partial\theta}{\partial\phi}=0 \quad \text{at } \phi=0 \text{ and for } 1\le R\le S_l \qquad (3b)$$

$$\frac{\partial\theta}{\partial\phi}=0 \quad \text{at } \phi=\pi/2 \text{ and for } 1\le R\le S_t \qquad (3c)$$

$$\frac{\partial\theta}{\partial N}=0 \quad \text{at } R=R_t \text{ and for } 0\le\phi\le\pi/2 \qquad (3d)$$

where N and R_t are the normal direction at the tip and dimensionless tip distance from the tube center respectively. The tip distance r_t for staggered and inline arrangement of tubes can be written as follows:

$$R_t = r_t/r_i = \begin{cases} \begin{bmatrix} S_t^(1+R_a^2)/\{2R_a\cos\phi(1+\tan\phi\tan\alpha_0)\} & \text{for} \quad 0 \le \phi \le 2\alpha_0 \\ S_t/\sin\phi & \text{for} \quad 2\alpha_0 \le \phi \le \pi/2 \end{bmatrix} & \text{staggered} \\ \begin{bmatrix} S_tR_a/\cos\phi & \text{for} \quad 0 \le \phi \le \alpha_0 \\ S_t/\sin\phi & \text{for} \quad \alpha_0 \le \phi \le \pi/2 \end{bmatrix} & \text{inline} \end{cases} \tag{4}$$

where

$$\begin{bmatrix} \alpha_0 \\ R_a \end{bmatrix} = \begin{bmatrix} \tan^{-1}(1/R_a) \\ S_l/S_t \end{bmatrix} \tag{5}$$

Eq. (1) along with boundary conditions (3a)–(3c) can be solved by separation of variables (Zabronski, 1955; Kundu and Das, 1999; Kundu, 2007) to get the following temperature distribution in the fin.

$$\theta = \frac{I_0(Z_0R)}{I_0(Z_0)} + \sum_{j=0}^{\infty} C_j\cos(2j\phi)\left[\frac{I_{2j}(Z_0)K_{2j}(Z_0R) - I_{2j}(Z_0R)K_{2j}(Z_0)}{I_{2j}(Z_0)}\right] \tag{6}$$

The coefficient C_j may be obtained by exploiting the remaining boundary condition (3d). Here, it may be noted that the conditions (3a)–(3c) have been satisfied exactly along the respective boundaries but the boundary condition (3d) cannot be matched exactly at the tip for the typical shape. This boundary condition has been satisfied at a large number of discrete points at the tip until a desired accuracy is reached. For the last boundary condition of the fin, the following mathematical expressions are obtained for staggered and inline arrangement of tubes:

After getting the temperature distribution in the fin, the heat transfer rate (q) through the symmetric section of the fin can be calculated as follows:

$$Q = \frac{q}{kr_i\pi(T_b - T_a)} = T[C_0 - Z_0I_1(Z_0)]/I_0(Z_0) \tag{11}$$

Flat fin efficiency is defined as the ratio of the rate of actual heat transfer rate Q through the flat fin to the rate of ideal heat transfer q_i if the entire fin surface were at its base temperature. Therefore, dimensionless Q_i can be calculated from the expression given below.

$$Q_i = \frac{q_i}{kr_i\pi(T_b - T_a)} = \begin{cases} Z_0^2T[R_aS_t^2/\pi - 1/2] & \text{staggered} \\ Z_0^2T[2R_aS_t^2/\pi - 1/2] & \text{inline} \end{cases} \tag{12}$$

Thus fin efficiency

$$\eta = Q/Q_i \tag{13}$$

Flat fin effectiveness is defined as the ratio of the rate of actual heat transfer through the flat fin to that which would be transferred (q_e) through the same base surface in the absence of the flat fin.

$$\left[\frac{\partial\theta}{\partial N}\right]_{R=R_t} = 0 = \begin{cases} \begin{bmatrix} \cos(\alpha_0-\phi) & \sin(\alpha_0-\phi) \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \partial\theta/\partial R \\ \partial\theta/R\partial\phi \end{bmatrix}_{R=R_t} & \begin{matrix} 0 \le \phi \le 2\alpha_0 \\ 2\alpha_0 \le \phi \le \pi/2 \end{matrix} & \text{staggered} \\ \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \partial\theta/\partial R \\ \partial\theta/R\partial\phi \end{bmatrix}_{R=R_t} & \begin{matrix} 0 \le \phi \le \alpha_0 \\ \alpha_0 \le \phi \le \pi/2 \end{matrix} & \text{inline} \end{cases} \tag{7}$$

Finally, the constants C_j are determined from Eqs. (6) and (7), and the following mathematical expressions are obtained.

$$Q_e = q_e/[kr_i\pi(T_b - T_a)] = (Z_0T)^2 \tag{14}$$

$$\frac{Z_0I_1(Z_0R_t)}{I_0(Z_0)} = \begin{cases} \begin{bmatrix} \sum_{j=0}^{\infty} C_j\cos(2j\phi)[Z_0B_1 + B_2\{1-\tan(2j\phi)\tan(\alpha_0-\phi)\}]/I_{2j}(Z_0) & \text{for} \quad 0 \le \phi \le 2\alpha_0 \\ \sum_{j=0}^{\infty} C_j\cos(2j\phi)[Z_0B_1 + B_2\{1-\tan(2j\phi)\cot\phi\}]/I_{2j}(Z_0) & \text{for} \quad 2\alpha_0 \le \phi \le \pi/2 \end{bmatrix} & \text{staggered} \\ \begin{bmatrix} \sum_{j=0}^{\infty} C_j\cos(2j\phi)[Z_0B_1 + B_2\{1-\tan(2j\phi)\tan\phi\}]/I_{2j}(Z_0) & \text{for} \quad 0 \le \phi \le \alpha_0 \\ \sum_{j=0}^{\infty} C_j\cos(2j\phi)[Z_0B_1 + B_2\{1-\tan(2j\phi)\cot\phi\}]/I_{2j}(Z_0) & \text{for} \quad \alpha_0 \le \phi \le \pi/2 \end{bmatrix} & \text{inline} \end{cases} \tag{8}$$

where

$$B_1 = I_{2j}(Z_0)K_{2j+1}(Z_0R_t) + I_{2j+1}(Z_0R_t)K_{2j}(Z_0) \tag{9}$$

and

$$B_2 = 2j\lfloor I_{2j}(Z_0R_t)K_{2j}(Z_0) - I_{2j}(Z_0)K_{2j}(Z_0R_t)\rfloor/R_t \tag{10}$$

The family of simultaneous equations given by Eq. (8) are solved to obtain a finite number (say p) values of C_j (say C_0 to C_p). The number p should be chosen in such a way so that the solution yields results of a desired accuracy. Then the temperature profile in the fin can easily be determined by substituting these unknowns in Eq. (6).

Thus fin effectiveness can be written as

$$\varepsilon = Q/Q_e \tag{15}$$

2.1. Sector method

In the sector method, the symmetric heat transfer module (shown in Fig. 2) of the flat fin is divided into a number of sectors. Each of the sectors, extending from the outer radius of the tube to the fin tip, subtends a small angle at the center of the tube and is separated from the neighboring sectors by imaginary insulating radial lines. The efficiency of the n-th sector having an area A_n is η_n, the efficiency of the entire flat fin can be expressed as follows (Shah, 1985):

$$\eta = \frac{\sum_{n=1}^{\infty}\eta_n A_n}{\sum_{n=1}^{\infty}A_n} \tag{16}$$

where

$$\eta_n = 2D_1/D_2 Z_0(R_{tn}^2 - 1) \tag{17}$$

$$D_1 = I_1(Z_0R_{tn})K_1(Z_0) - I_1(Z_0)K_1(Z_0R_{tn}) \tag{18}$$

$$D_2 = I_0(Z_0)K_1(Z_0R_{tn}) + I_1(Z_0R_{tn})K_0(Z_0) \tag{19}$$

$$A_n = 2a_n/r_i^2 = \Delta\phi(R_{tn}^2 - 1) \tag{20}$$

$$R_{tn} = \begin{cases} \begin{bmatrix} S_t(1+R_a^2)/\{2R_a\cos\phi_n[1+\tan\phi_n\tan\alpha_0]\} & \text{for } 0 \le \phi_n \le 2\alpha_0 \\ S_t/\sin\phi_n \quad \text{for} \quad 2\alpha_0 \le \phi_n \le \pi/2 \end{bmatrix} & \text{staggered} \\ \begin{bmatrix} S_tR_a/\cos\phi_n & \text{for} \quad 0 \le \phi_n \le \alpha_0 \\ S_t/\sin\phi_n & \text{for} \quad \alpha_0 \le \phi_n \le \pi/2 \end{bmatrix} & \text{inline} \end{cases} \tag{21}$$

$$\Delta\phi = \pi/(2m) \tag{22}$$

and

$$\phi_n = (2n-1)\Delta\phi/2 \tag{23}$$

m is the total number of the sectors considered.

After getting flat fin efficiency, flat fin effectiveness can be obtained by the sector method from the expression given below:

$$\varepsilon = Q_i\eta/(Z_0T)^2 \tag{24}$$

2.2. Optimization

Dimensionless fin volume of the symmetric heat transfer module can be written as follows:

$$U = \frac{v}{2r_i^3} = T(R_aS_t^2/F - \pi/4) \tag{25}$$

where F is a constant, 2 for staggered arrangement and 1 for inline arrangement.

For the present case it is assumed that the tube radius r_i, fin material thermal conductivity k and the convective heat transfer coefficient h are specified, i.e. the Biot number is known. Then the rate of heat transfer and the plate volume are functions of plate thickness T, pitch ratio R_a and S_t. The optimum fin dimension may be obtained by maximizing heat transfer rate for a given fin volume or minimizing the fin volume for a given heat duty. Therefore, it is an optimization problem having three variables and one constraint. The optimality criterion may be determined by the Lagrange multiplier technique. Eliminating the Lagrange multiplier from the system of equations, one gets the generalized condition of optimality as follows (Stoecker, 1989):

$$\begin{bmatrix} \partial Q/\partial T & -\partial U/\partial T \\ \partial Q/\partial R_a & -\partial U/\partial R_a \end{bmatrix} \begin{bmatrix} \partial U/\partial S_t \\ \partial Q/\partial S_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{26}$$

Eq. (26) can be written by using Eqs. (11) and (25) as

$$\begin{bmatrix} f_1(T,R_a,S_t) \\ f_2(T,R_a,S_t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} S_tR_a(2C_0 + 2T\partial C_0/\partial T + g) - (R_aS_t^2 - F\pi/4)\partial C_0/\partial S_t \\ 2R_a\partial C_0/\partial R_a - S_t\partial C_0/\partial S_t \end{bmatrix} \tag{27}$$

$$g = Z_0^2I_2(Z_0) + Z_0I_1(Z_0)/I_0(Z_0) - Z_0^2I_1(Z_0)/I_0(Z_0) \tag{28}$$

To get the optimum dimensions, Eq. (27) is to be solved along with Eq. (25) if the design problem involves constant volume. Otherwise, if the required heat duty is specified, Eq. (27) is to be solved simultaneously with Eq. (11). Therefore, in either case, two nonlinear equations are to be solved simultaneously to get three geometric parameters of the irregular flat fin. This can be done applying the generalized Newton–Raphson method (Scarborough, 1966). The following steps are required to obtain the optimum dimensions by using Newton–Raphson method:

The design constraint either heat transfer duty or fin volume can be expressed by a function as

$$f_3(T,R_a,S_t) = 0 = \begin{cases} T[C_0 - Z_0I_1(Z_0)]/I_0(Z_0) - Q \\ T(R_aS_t^2/F - \pi/4) - U \end{cases} \tag{29}$$

From Eqs. (27) and (29), the optimum dimensions can be approximated by using the generalized Newton–Raphson method:

$$\begin{bmatrix} T_{j+1} \\ R_{aj+1} \\ S_{tj+1} \end{bmatrix} = \begin{bmatrix} T_j \\ R_{aj} \\ S_{tj} \end{bmatrix} - \mathbf{J}(T_jR_{aj},S_{tj})^{-1} \begin{bmatrix} f_1(T_j,R_{aj},S_{tj}) \\ f_2(T_j,R_{aj},S_{tj}) \\ f_3(T_j,R_{aj},S_{tj}) \end{bmatrix} \tag{30}$$

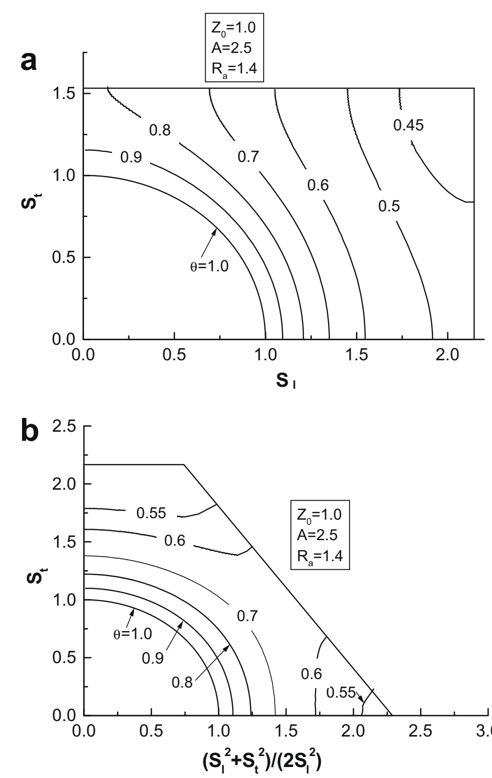

Fig. 3. Isotherms of an irregular polygonal fin: (a) inline arrangement; and (b) staggered arrangement.

where **J** is the Jacobian matrix expressed as

$$\mathbf{J}(T_j, R_{aj}, S_{tj}) = \begin{bmatrix} \partial f_1/\partial T & \partial f_1/\partial R_a & \partial f_1/\partial S_t \\ \partial f_2/\partial T & \partial f_2/\partial R_a & \partial f_2/\partial S_t \\ \partial f_3/\partial T & \partial f_3/\partial R_a & \partial f_3/\partial S_t \end{bmatrix} \tag{31}$$

The subscript "j" denotes the value of jth iteration. The convergence criteria at each step of the iteration must be satisfied in the following equation.

$$\mathrm{Max}\{\Delta_1, \Delta_2, \Delta_3\} < 1 \tag{32}$$

where the expressions for Δ_1, Δ_2 *and* Δ_3 are given by

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \begin{bmatrix} |\partial \Gamma_1/\partial T|_j + |\partial \Gamma_2/\partial T|_j + |\partial \Gamma_3/\partial T|_j \\ |\partial \Gamma_1/\partial R_a|_j + |\partial \Gamma_2/\partial R_a|_j + |\partial \Gamma_3/\partial R_a|_j \\ |\partial \Gamma_1/\partial S_t|_j + |\partial \Gamma_2/\partial S_t|_j + |\partial \Gamma_3/\partial S_t|_j \end{bmatrix} \tag{33}$$

$$\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{bmatrix} = \begin{bmatrix} T - \mathrm{Det}\,\mathbf{\Omega_1}/\mathrm{Det}\mathbf{J} \\ R_a - \mathrm{Det}\,\mathbf{\Omega_2}/\mathrm{Det}\mathbf{J} \\ S_t - \mathrm{Det}\,\mathbf{\Omega_3}/\mathrm{Det}\mathbf{J} \end{bmatrix} \tag{34}$$

$$\mathbf{\Omega_1} = \begin{bmatrix} f_1 & \partial f_1/\partial R_a & \partial f_1/\partial S_t \\ f_2 & \partial f_2/\partial R_a & \partial f_2/\partial S_t \\ f_3 & \partial f_3/\partial R_a & \partial f_3/\partial S_t \end{bmatrix} \tag{35}$$

$$\mathbf{\Omega}_2 = \begin{bmatrix} \partial f_1/\partial T & f_1 & \partial f_1/\partial S_t \\ \partial f_2/\partial T & f_2 & \partial f_2/\partial S_t \\ \partial f_3/\partial T & f_3 & \partial f_3/\partial S_t \end{bmatrix} \tag{36}$$

and

$$\mathbf{\Omega}_3 = \begin{bmatrix} \partial f_1/\partial T & \partial f_1/\partial R_a & f_1 \\ \partial f_2/\partial T & \partial f_2/\partial R_a & f_2 \\ \partial f_3/\partial T & \partial f_3/\partial R_a & f_3 \end{bmatrix} \tag{37}$$

The above procedures are repeated till the geometrical roots T, R_a and S_t are obtained to a desired accuracy (10^{-6} in the present study). In this connection, it may be noted that the root finding algorithm requires the values of different first, second and cross derivatives of C_0, namely, $\partial C_0/\partial T$, $\partial C_0/\partial R_a$, $\partial C_0/\partial S_t$, $\partial^2 C_0/\partial T^2$, $\partial^2 C_0/\partial R_a^2$, $\partial^2 C_0/\partial S_t^2$, $\partial^2 C_0/\partial T \partial R_a$, $\partial^2 C_0/\partial R_a \partial S_t$ *and* $\partial^2 C_0/\partial S_t \partial T$. They can be obtained by successive partial differentiation of Eq. (8) and then they have been solved by Gauss Elimination method. Finally, it may be noted that the initial guess value of the optimum root for the Newton–Raphson method has been chosen in such a way that the convergence criterion of (Scarborough, 1966) has been satisfied.

3. Results and discussion

Based on the above analysis, results are taken for the variation of different design variables. At first fin temperature is determined by the semi-analytical method and a temperature contour is plotted over the symmetric sector of inline and staggered arrays of tubes for the same surface area and thermo-geometric parameters.

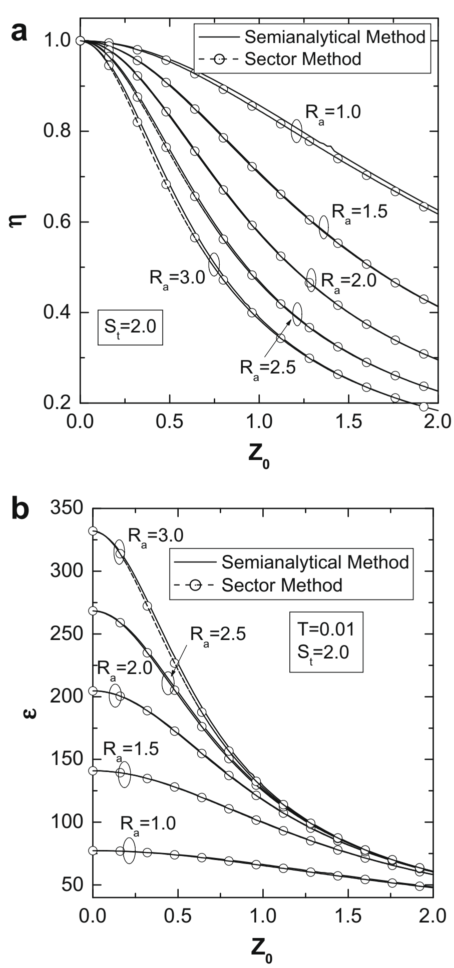

Fig. 4. Comparison of results for performances of plate fins with a staggered array of tubes predicted by the present semi-analytical and sector methods: (a) plate efficiency; and (b) plate effectiveness.

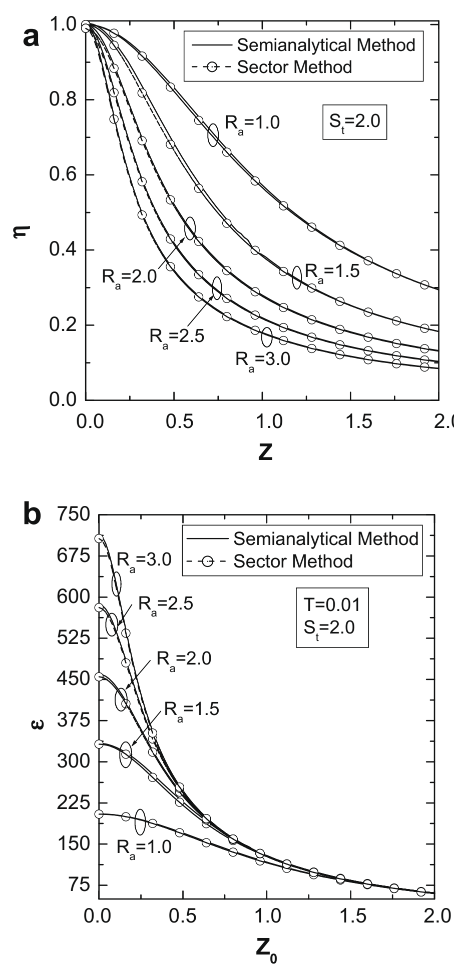

Fig. 5. Comparison of results for performance of plate fins with an inline array of tubes predicted by the present semi-analytical and sector methods: (a) plate efficiency and (b) plate effectiveness.

The surface area for both the heat transfer module can be expressed in dimensionless form as

$$A = a/2r_i^2 = S_t^2 R_a/F - \pi/4 \tag{38}$$

Fig. 3 depicts the aforementioned variation. A concentric contour is obtained near the tube region of the plate which indicates almost one-dimensional heat conduction occurred in this region of the plate. As the radial distance of the plate from the centre of the tube increases, the two-dimensional heat conduction in the plate gradually dominates. This effect becomes a maximum at the tip boundary of the symmetric sector. In comparison with the inline array of tubes, two-dimensional temperature field in the plate circumscribing staggered circular tubes is exhibited more due to the geometric variation of the symmetric sectors. In addition from the figure, it can also be noted that the temperature variation in the symmetric module of the plate in the radial direction decreases sharply in the case of inline array of tubes because more resistance occurs to transfer heat due to conduction.

Now, an attention has been given to determine the performance of a tube-and-fin exchanger by the present semi-analytical and sector methods. In a tube-and-fin heat exchanger, staggered or inline tubes of array are commonly employed in practical applications. The performance of this type of heat exchangers is mainly dependent upon the performance of fins attached with the tubes. The fin performance, namely, fin efficiency and fin effectiveness depends upon the thermophysical and geometric parameters. Fig. 4 depicts the fin performance as a function of Z_0 and R_a predicted by the semi-analytical model and the sector method. In order to establish the accuracy level of the sector method, different fin geometries have been considered with the variation of R_a. The fin efficiency and fin effectiveness obtained from both the methods decrease with the increase in Z_0. The increment of R_a gives a decreasing fin efficiency, whereas, a reversed trend is observed with the effect of R_a on the fin effectiveness. It is an expected observation due to increase in surface area with R_a. In comparison of methods of prediction of results, an approximate sector method predicts a reasonably matching result with that predicted by present semi-analytical technique. In every situation of results, the sector method gives an under predict value. Since heat flow seeks the path of least resistance, this idealization of ideal radial heat flow may result in the value of the fin efficiency and fin effectiveness by sector method given by Eqs. (16) and (24) respectively, lower than the actual, a conservative value. The same nature is obtained for the inline array of tubes predicted by present semi-analytical and sector methods which is displayed in Fig. 5.

Next a comparative study has been made for fin performances of inline and staggered arrays of tubes for the same surface area, thermophysical and geometric parameters. Fig. 6 is drawn for the

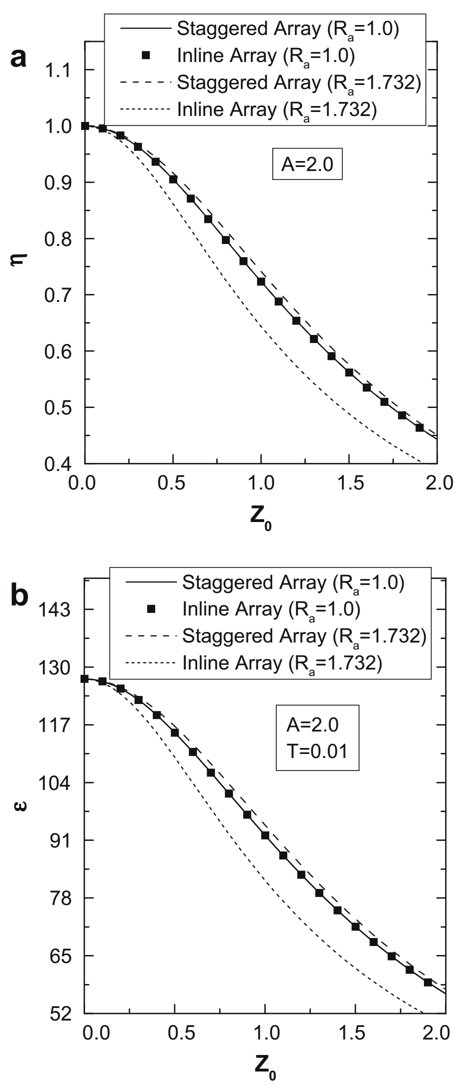

Fig. 6. Comparison of fin performance for staggered and inline arrays of tubes: (a) fin efficiency and (b) fin effectiveness.

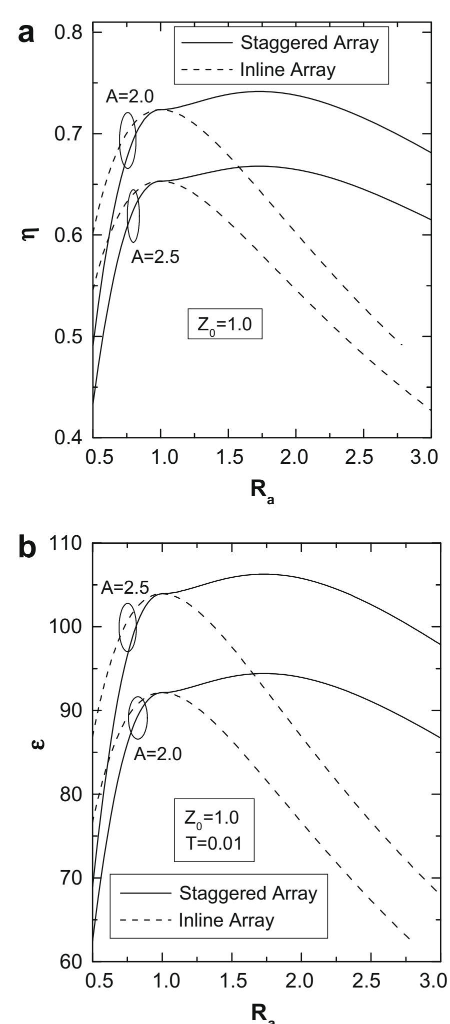

Fig. 7. Fin performances for staggered and inline arrays of tubes as a function of R_a: (a) fin efficiency and (b) fin effectiveness

aforementioned purpose. From the graph, it can be demonstrated that the fin efficiency of staggered array of tubes is high or lower in comparison with that of inline array of tubes depending upon the value of R_a selected. For R_a value greater than one, for the same surface area, thermophysical and geometric parameters, temperature variation in the plate from base to tip for staggered array of tubes diminishes at a slower rate than that in the plate for inline array of tubes as already shown in Fig. 3. This is occurred due to less conductive resistance in the plate in the case of staggered arrays which increases the fin performance. The fin performance is identical for inline and staggered arrays of tube for $R_a = 1$ because both the arrays are converted into an equivalent symmetric sector. In general for a constant R_a, fin performance for both the arrays diminishes with the increase in Z_0. On the other hand, a comparative value of fin performance between inline and staggered arrays of tubes not only depends upon the fin geometric parameter R_a but also is a function of fin parameter Z_0. For the lower value of Z_0, and for R_a greater than one, the fin performances of the inline array of tubes gives a slightly lower value with respect to that for the staggered array of tubes. With the increase in Z_0, this difference in fin performances increases. Therefore, form the figure, it can be concluded that whether the difference in fin performances between staggered and inline arrays of tubes is more or less, it depends on the magnitude of Z_0 and R_a associated with the design under a constant surface area A.

The effect of R_a on the fin performances is shown in Fig. 7. During this investigation, the fin parameter Z_0 and surface area A are taken constants. For a constant surface area A, the fin performance increases first with the increase in R_a , reaches to a maximum value for a particular R_a and then decreases with the further increase in R_a. This trend has been observed for both the inline and staggered arrays of tubes. The nature of trend has been unaltered for different constant fin surface areas taken. The increase in area both decreases the fin efficiency and increases the fin effectiveness. From the figure, it can be highlighted that fin performances are not only dependent upon the array of tubes but also it depends strongly upon the geometrical parameter R_a. For a lower value R_a less than unity, fin performance of inline array of tubes gives a higher value. The performance for both the arrays gives the same value for $R_a = 1$. The fin performance of the staggered array is always higher in comparison to that of the inline array for $R_a > 1$. For the inline array of tubes, the maximum fin performance is obtained at $R_a = 1$ whereas maximum fin performance for the staggered array of tubes is noticed at $R_a = 1.732$. The difference in fin performances of staggered and inline arrays of tubes increases significantly with the increase in R_a due to decrease in fin performances rapidly in the case of inline array of tubes. From this figure, it can be concluded that a higher fin performance from either inline or staggered array of tubes are obtained depending upon the magnitude of R_a selected for a design application.

The optimization study has been carried out in a generalized manner such that all the dimensions are varied to obtain an opti-

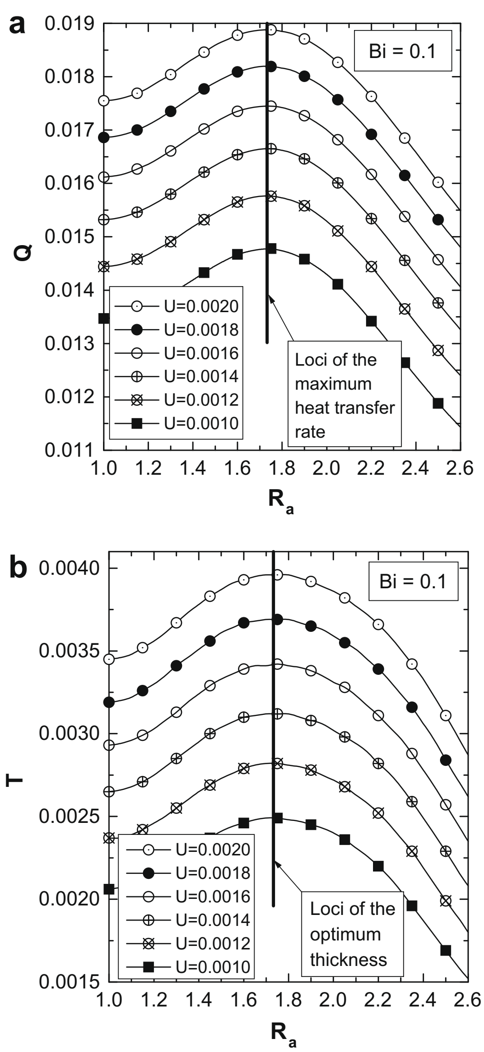

Fig. 8. Optimum design of plate fins as a function of R_a for staggered array of tubes: (a) heat transfer rate and (b) plate thickness.

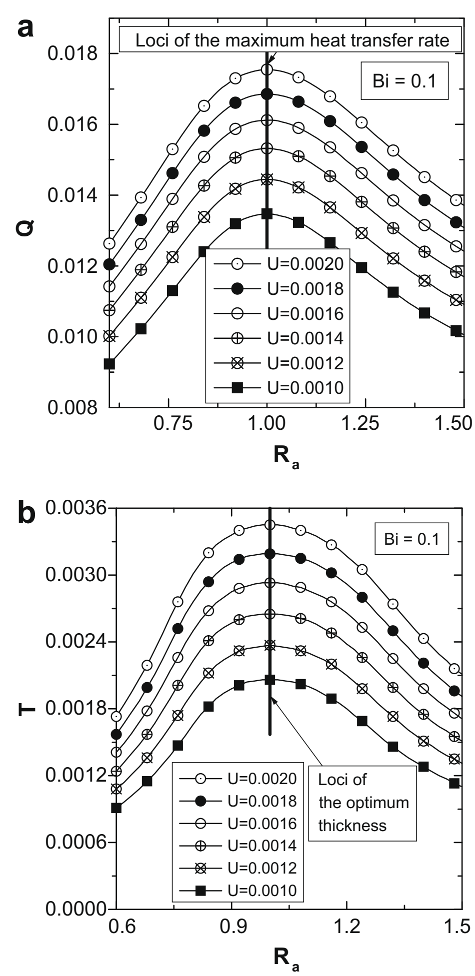

Fig. 9. Optimum design of plate fins as a function of R_a for inline array of tubes: (a) heat transfer rate and (b) plate thickness.

mum geometry with satisfying either maximization of heat transfer rate for a given fin volume or minimization of fin volume for a given heat transfer duty. Thus, the optimum condition is dependent upon the fin geometric parameters R_a, S_l and T, thermogeometric parameter Bi, and a constraint condition either fin volume or heat transfer duty adopted. In the present problem, thermogeometric variable Bi is a known. Therefore, the optimum design condition is a function of fin geometric parameters only. The optimality criteria are derived by using Euler equations after elimination of Lagrange multiplier and two optimality conditions are achieved. These two equations can be solved simultaneously with the constraint equation. The constraint equation either heat transfer duty or fin volume can be taken depending upon the design requirement. In this connection, it may be noted that, in the previous work, the geometric parameter R_a had not been considered variable for the analysis of plate fin heat exchangers. The R_a value had been selected a constant hypothetically for the inline and staggered arrays of tubes. In the present study, R_a is taken a variable and a methodology has been suggested to determine the optimum R_a. Fig. 8 is drawn with the variation of heat transfer rate and plate thickness for staggered array of tubes as a function of R_a for different fin volume and a constant Bi. For every fin volume, there is an optimum R_a for which heat transfer rate through the plate becomes a maximum as well as plate thickness reaches to a maximum value. However, this optimum R_a does not depend upon the magnitude of fin volume. From this observation, it can be highlighted that with the increase in fin volume, the sizes of the optimum fin shape increases. From the optimum thickness, it can be highlighted that the fin thickness becomes maximum at the optimum point for a given fin volume due to symmetric sector converted into a smallest one satisfying a minimum thermal resistance. However, the geometric parameter R_a is unchanged but the optimum fin thickness increases monotonically. The same observation has been found in the case of inline arrangement of tubes as shown in Fig. 9. Thus from these Figs. 8 and 9, it can be demonstrated that the optimum R_a is only dependent upon the arrangement of tubes, not any other design parameters.

From Figs. 8 and 9, it is once more mentioned that the optimum R_a for the inline and staggered arrays of tubes are a constant irrespective of constraint values (U or Q) adopted. Hence in the design aspect, the optimization analysis of plate fins is not carried out to determine the optimum R_a. In practical applications, the maximum number of tubes may be fitted with the plate in a shell side in which the shell dimensions are being kept at a constant. In that situations, the geometry is required to arrange in such a way that the plate geometric parameter R_a may not always be possible to design at its optimum value. In that applications, it is required to optimize the fin dimensions with the consideration of additional constraint R_a. Therefore, the present analysis is focused to study on optimization of plate fins in a fin-and-tube heat exchanger for a specified R_a. For this observation, Fig. 10 is plotted for the staggered array of tubes. Fig. 10 depicts the variation of optimum parameters with fin volume for different R_a values. Among the different R_a selected for plotting, the optimum R_a has also been taken. The variation of maximum heat transfer rate with fin volume is shown in Fig. 10a. From this figure, it can be mentioned that for the optimum

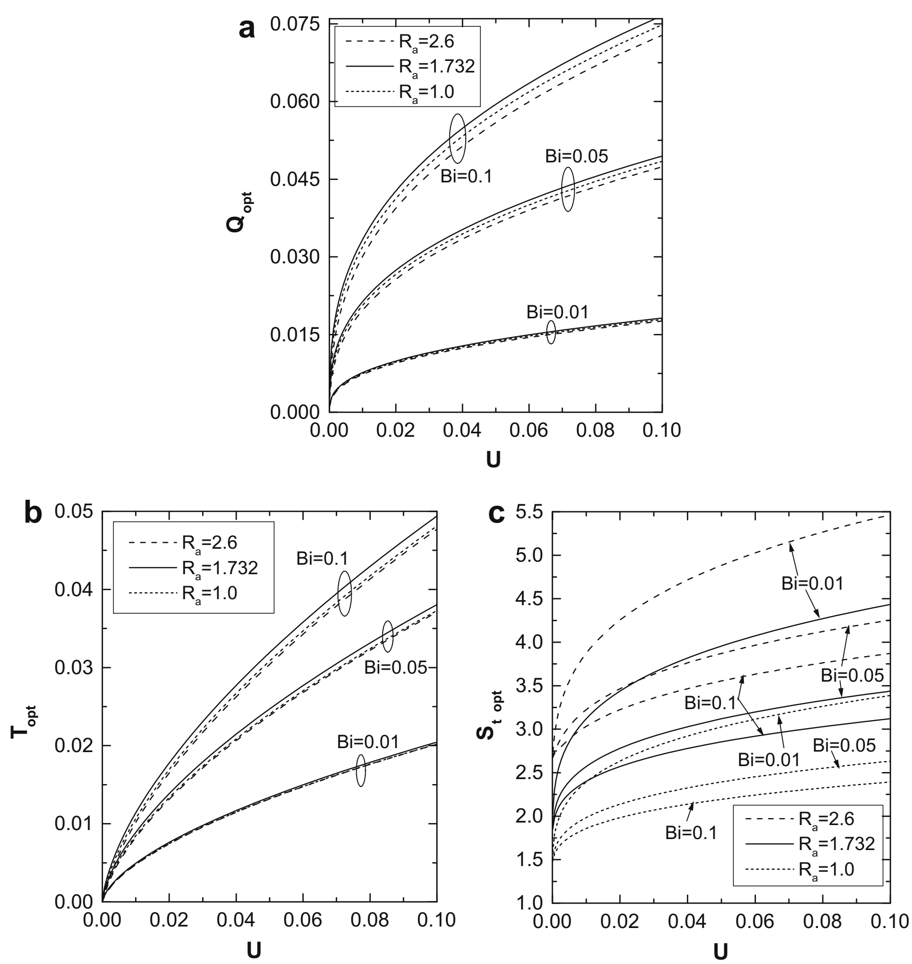

Fig. 10. Optimum design of plate fins for staggered array of tubes: (a) optimum heat transfer rate, (b) optimum plate thickness and (c) optimum transverse length.

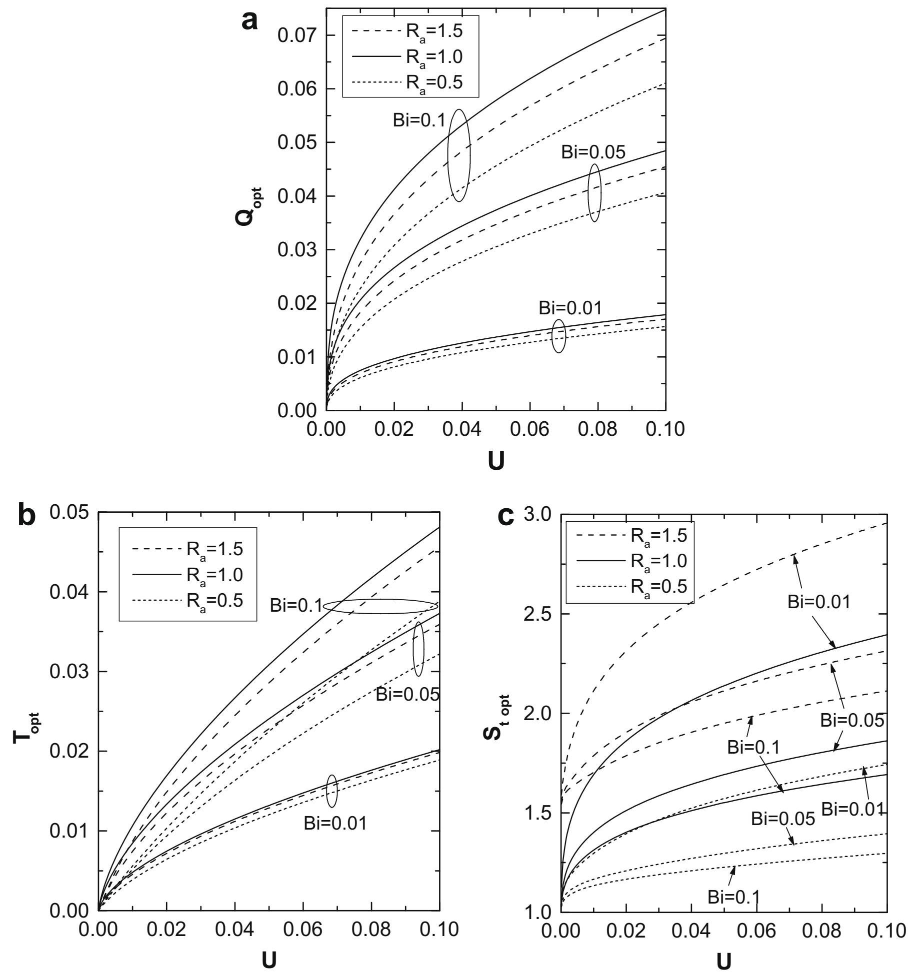

Fig. 11. Optimum design of plate fins for inline array of tubes: (a) optimum heat transfer rate, (b) optimum plate thickness and (c) optimum transverse length.

R_a value, maximum heat transfer rate is noticed for a constant fin volume. From the variation of R_a from its optimum value, the optimum heat transfer rate decreases. Again the optimum heat transfer rate depends significantly upon the Bi. For a higher Bi, the optimum heat transfer rate is also higher. From the graph, it can also be highlighted that the difference in heat transfer rate for different R_a is dependable upon the magnitude of Bi used. It is of interest to mention that a greater difference is noticed for higher values of Bi. For every situation, the optimum heat transfer rate increases with the increase in fin volume. Fig. 10b depicts the variation of optimum plate thickness with the plate volume. From the optimum plate thickness, it may be noted that for a given Bi and fin volume, the plate geometry for the maximum heat transfer rate requires a higher plate thickness. In addition the optimum thickness increases with the Bi. The variation of optimum S_t with fin volume is depicted in Fig. 10c for different R_a. The same trend in results for inline array of tubes for the variation of optimum design variables with the fin volume adopted for an additional constraint R_a is obtained and it is shown in Fig. 11.

4. Conclusions

The performance and optimization analysis of plate fins circumscribing circular tubes for inline and staggered arrays of tubes are demonstrated by a semi-analytical technique. The sector method is also applied on the same problem to determine the fin performance. The result obtained by using the sector method is closer to that values obtained by semi-analytical technique. However, the sector method always predicts lower values of fin performances. In comparison of fin performances for staggered and inline arrays of tubes, fin performance for the staggered geometry is always higher for R_a value greater than one and on the other hand, a higher fin performance for a constant surface area is not only dependent upon the arrangement of tubes but also depends upon the magnitude of fin parameter Z_0 and geometric parameter R_a. With the increase in R_a for both the arrays, fin performance increases first, reaches to a maximum for a particular R_a and then decreases.

It may be noted that the present adopted optimization scheme is in the most generalized form as it does not only take care of either volume or heat duty constraint, it is also applicable to both regular and irregular polygonal fins. If the optimization equations are solved in their present form, the unique values of R_a ($=S_l/S_t$) is obtained for inline and staggered arrangement of tubes so that the heat transfer rate is maximum for any given fin volume. It is interesting to note that the solution yields $R_a = 1$ for the inline array of tubes and $R_a = \sqrt{3}$ for the staggered array. This implies that a square array for inline tubes and an equilateral triangular array for staggered arrangement results in maximum heat transfer irrespective of fin volume and thickness. This result, though intuitively well known for long, so far it has not been proved mathematically.

If the optimum dimensions are to be found for irregular polygonal fins, R_a is to be specified *a priori*. The value of R_a may be fixed if the size of the heat exchanger envelope is specified or from other practical limitations. For a given value of R_a, unique optimum values of S_t and T may be obtained depending upon the fin volume.

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